

Surfaces of the Reconstruction Error of Gaussian Fields with Non-stationary Regions

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Abstract. This paper considers the reconstruction error of Gaussian fields, when a non-stationary region is present. The non-stationary character is only given for spatial data but not for spatio-temporal ones. The reconstruction method is specified on the basis of the conditional mean rule when the quantity of samples is restricted. The method estimates the surface of the optimal reconstruction error functions in the whole space domain of fields. The ability of the method is demonstrated with a simple statistical description of fields on stationary or non-stationary regions. The Gaussian fields are mainly described by a spatial covariance function to describe the nature of Gaussian Markov fields.

Keywords: Gaussian markov fields, mean square reconstruction error surfaces, stationary and non-stationary fields.

1 Introduction

The description of the Sampling-Reconstruction Procedure (SRP) of random fields has been discussed in many publications [1-7]. It is noticed in the majority of the works [1-4] that the probability density function (pdf) of fields is not mentioned. Therefore, this situation is the source of many complexities and imprecision in many papers devoted to SRP of fields.

The problem of the estimation of the error reconstruction in the SRP of random fields has practically not been solved in two-dimensions yet. One can appreciate that the reconstruction error function has been determined in [3, 4] only for a one-dimensional case, hence the reconstruction error of random processes is defined instead of the reconstruction error of random fields; and only some bounds for the reconstruction error has been evaluated in [5, 6].

In the present paper, a generalization of the conditional mean rule [8] is proposed for the statistical description of SRP of Gaussian random fields. With this rule the estimation of an unknown random variable is specified by the conditional mathematical expectation. This method provides the minimum of the mean square reconstruc-

tion error. The description of various kinds of random processes and some Gaussian fields has been determined by this rule in [9-11].

The reconstruction problem of continuous random fields is described, in this work, only in the non-stationary space, that is, when they do not depend on time. In order to describe the Gaussian fields completely we use the usual statistical characteristics – the spatial covariance function and the mathematical expectation. In this work, the Gaussian random fields are specifically defined by the non-stationary spatial exponential covariance function. Non-stationary random fields are required in modelling image with complex patterns, in geophysical and environmental applications. For these purposes is common to describe spatial data with non-stationary covariance structure.

There has been little progress in the problem of non-stationary fields. Some works have been focused to transform non-stationary fields into stationary ones [12, 13].

The central differences between our methodology and that presented by the previous literature are the following: a) the conditional mean rule is applied to describe the reconstruction procedure of random fields; b) the optimal SRP algorithms are analyzed with an arbitrary and limited number of samples; c) the nature of the multi-dimensional pdf of fields is determined by the Gaussian distribution; d) the stationary or non-stationary character of Gaussian fields is described in any spatial region with these properties; e) the minimum reconstruction error surfaces are evaluated on the whole space domain.

2 Description of method

Generally, random fields $F(x,y,t)$ are considered as a continuous three-dimensional stochastic process with space variables (x,y) and the time variable t . This field can be completely described by knowledge of its joint pdf

$$p(F_1, F_2, \dots, F_N; x_1, y_1, t_1, x_2, y_2, t_2, \dots, x_N, y_N, t_N), \quad (1)$$

for all sample points N , where (x_i, y_i, t_i) indicate space and time samples of random fields.

The field is represented by infinity of surfaces as separate realizations. Generally, it is difficult to know the high-order joint probability densities of the fields. But, in the Gaussian case they can be completely determined by its mathematical expectation $\langle F(x,y,t) \rangle = m(x,y,t)$ and its covariance function $K(x_2-x_1, y_2-y_1, t_2-t_1)$, where $\langle \cdot \rangle$ is the expectation operator.

When we fix an arbitrary set of N samples of the field $S = \{F(x_1, y_1, t_1), F(x_2, y_2, t_2), \dots, F(x_N, y_N, t_N)\}$, the random field is considered as a conditional field where all its realizations pass through all fixed points of the set S . The general expressions for the conditional mean matrix and the conditional covariance matrix of Gaussian multidimensional random variable are known [8].

In a variety of systems, the spatial and time domains of the fields are separable so that the covariance function may be written as

$$K(x_2-x_1, y_2-y_1, t_2-t_1) = K_{xy}(x_2-x_1, y_2-y_1) K_t(t_2-t_1). \quad (2)$$

The non-stationary spatial covariance function is determined as a function of the following parameters: the difference in each spatial coordinates, x_2-x_1 and y_2-y_1 ; and the variation of the inverse of covariance radius both on x -axis and y -axis, which are represented by ξ_x or ξ_y , correspondingly. Moreover, it is possible to define the beginning of the non-stationary region of the field on x -axis and y -axis by L_x and L_y .

With the above-mentioned information the non-stationary covariance function has the following representation:

$$K_{xy}(x_2-x_1, y_2-y_1, \xi_x, \xi_y)=K_{nstat}(x, y). \quad (3)$$

Hence, when we apply the conditional mean rule to two-dimensional Gaussian fields, we have the following expressions:

$$\begin{aligned} \tilde{m}(x, y) = m(x, y) + \sum_{i=1}^N \sum_{j=1}^N K_{xy}(x-x_i, y-y_i, \xi_x, \xi_y) \times \\ \times a_{ij} [F(x_j, y_j) - m(x_j, y_j)], \end{aligned} \quad (4)$$

$$\begin{aligned} \tilde{\sigma}^2(x, y) = \sigma^2(x, y) - \sum_{i=1}^N \sum_{j=1}^N K_{xy}(x-x_i, y-y_i, \xi_x, \xi_y) \times \\ \times a_{ij} K_{xy}(x_j-x, y_j-y, \xi_x, \xi_y), \end{aligned} \quad (5)$$

where $\sigma^2(x, y)$ is the unconditional variance of the initial field and a_{ij} represents to each element of the inverse non-stationary covariance matrix:

$$\mathbf{A} = \mathbf{K}_{xy}^{-1}(x_i-x_j, y_i-y_j, \xi_x, \xi_y). \quad (6)$$

With (4) and (5), we can characterize the surfaces of the optimal reconstruction function and the minimum reconstruction error function of the fields.

Equations (4) and (5) may be useful to analyse stationary and non-stationary random fields if the covariance function is characterized with these properties.

3 Results

Now we employ the expressions (4) and (5) for some examples, where the quantity of samples is sixteen and their Cartesian coordinates are fixed as follows: $x_1=y_1=-1.5$; $x_2=-1.5$, $y_2=-0.5$; $x_3=-1.5$, $y_3=0.5$; $x_4=-1.5$, $y_4=1.5$; $x_5=-0.5$, $y_5=-1.5$; $x_6=y_6=-0.5$; $x_7=-0.5$, $y_7=0.5$; $x_8=-0.5$, $y_8=1.5$; $x_9=0.5$, $y_9=-1.5$; $x_{10}=0.5$, $y_{10}=-0.5$; $x_{11}=0.5$, $y_{11}=0.5$; $x_{12}=0.5$, $y_{12}=1.5$; $x_{13}=1.5$, $y_{13}=-1.5$; $x_{14}=1.5$, $y_{14}=-0.5$; $x_{15}=1.5$, $y_{15}=0.5$; $x_{16}=1.5$ and $y_{16}=1.5$. These samples are arranged in a four by four grid.

In this case, the field is modelled by the exponential covariance function which describes the Markov Gaussian field. Its mathematical expression is:

$$K_{nstat}(x, y) = \sigma^2 \exp[-\xi_x |x_2-x_1| - \xi_y |y_2-y_1|], \quad (7)$$

In this paper, we consider that the exponential covariance function has the trend type of non-stationarity. Hence, the inverse of covariance radii is given by the following expressions:

$$\xi_x = \alpha_x + \gamma_x |x - L_x|, \text{ when } x \geq L_x, \quad (8)$$

$$\xi_y = \alpha_y + \gamma_y |y - L_y|, \text{ when } y \geq L_y, \quad (9)$$

where α_x and α_y are the initial values of the inverse of the covariance radius on x -axis and y -axis, respectively. In all examples we consider $\alpha_x = \alpha_y = 1$.

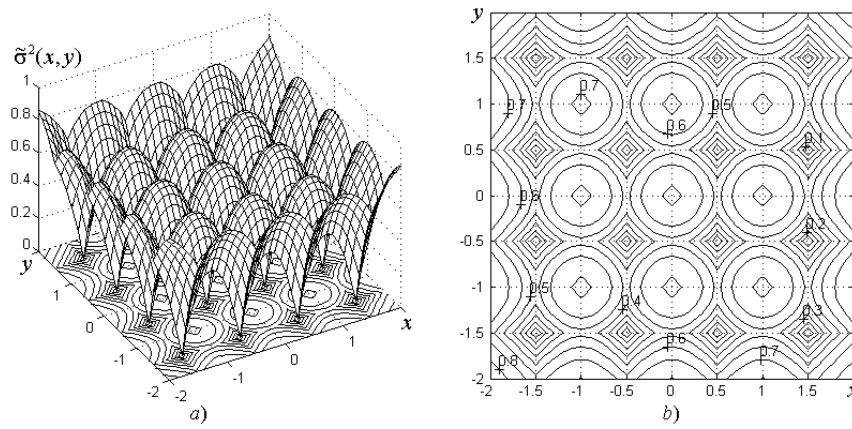


Fig. 1. Estimation of the reconstruction error of the stationary random field: *a)* the error reconstruction surface and *b)* its contour lines. Sixteen samples are spaced apart to an unitary distance.

The non-stationary character of the fields is limited by: $x \geq L_x$ and $y \geq L_y$. γ_x and γ_y are the variation factors of ξ_x and ξ_y on x -axis and y -axis. Therefore, the modified exponential covariance function is given by

$$K_{nstat}(x, y) = \sigma^2 \exp \left[-(\alpha_x + \gamma_x |x - L_x|) |x_2 - x_1| - (\alpha_y + \gamma_y |y - L_y|) |y_2 - y_1| \right]. \quad (10)$$

In the first example, the field is considered with a stationary character. For this case, the parameters γ_x and γ_y have the zero value in (9), and $m(x, y) = \langle F(x, y) \rangle = 0$ and the variance $\sigma^2(x, y) = \sigma^2 = 1$ in (4) and (5). The surface of the reconstruction error function $\sigma^2(x, y)$ of the field is shown in Fig. 1. In Fig. 1, we can observe that the reconstruction error function is zero at the location of spatial samples points (x_i, y_i) , because in these locations we know the exact value of the field realization.

However, the reconstruction error function has maximum errors at the centre of the interval, that is, among neighbouring samples. These maxima raise if the coordinates of the current samples are locate at a bigger distance. Due to the field realization pre-

sents uniform sampling and has a Markovian property and a stationary character, all maximum errors are exactly the same.

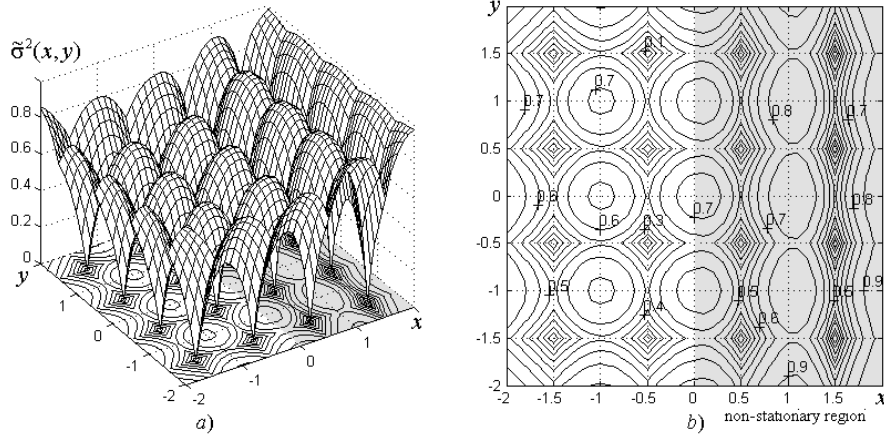


Fig. 2. Estimation of the reconstruction error of the random field when a non-stationary region is limited by the bound $x \geq L_x = 0$: a) the error reconstruction surface and b) its contour lines.

In the next instance, the field is considered as non-stationary on the x -axis and stationary on the y -axis. The variation factor γ_y defined here is zero, so the exponential covariance function depicted by (10) is reduced as follow

$$K_{nstat}(x, y) = \sigma^2 \exp \left[-(\alpha_x + \gamma_x |x - L_x|) |x_2 - x_1| - \alpha_y |y_2 - y_1| \right]. \quad (11)$$

The surface of the reconstruction error function $\tilde{\sigma}^2(x, y)$ of the field characterized by (11) is shown in Fig. 2. In this case, the field presents a non-stationary region in $x \geq L_x = 0$ and the variation factors are $\gamma_x = 0.25$ and $\gamma_y = 0$. We can notice that the maxima of the reconstruction error function increase when the value of the horizontal coordinate x tends to be far away of the limit L_x , due to the non-stationary behaviour of this region. However, if $x \leq L_x = 0$ the field maintains the stationary character.

In the last example, the non-stationary behaviour of the field affects the spatial structure of the field on the x -axis.

In Fig. 3 the surface of the reconstruction error function is given when the non-stationary region is contained into $x \geq L_x = -0.5$ and $y \geq L_y = -0.5$ and the variation factors of α_x and α_y have the same value: $\gamma_x = \gamma_y = 0.25$. It is important to notice that the maximum of the reconstruction error is bigger when the spatial coordinates of field are far away from the bounds L_x and L_y .

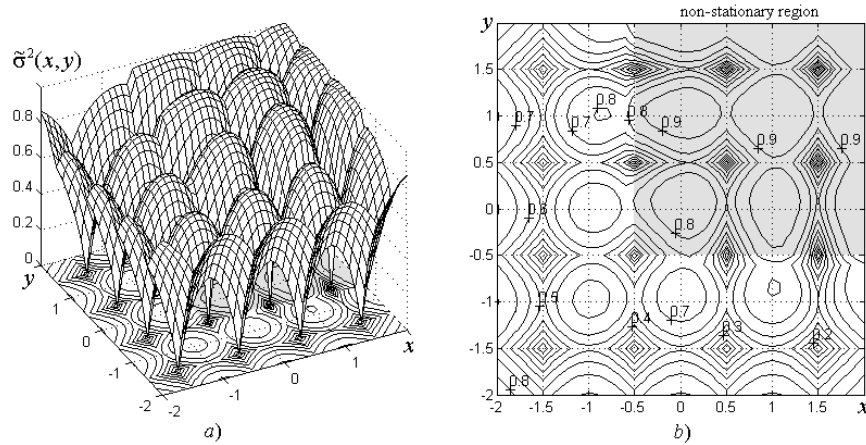


Fig. 3. Estimation of the reconstruction error of the random field when a non-stationary region is limited by $x \geq L_x = -0.5$ and $y \geq L_y = -0.5$: a) The error reconstruction surface and b) its contour lines.

4 Conclusions

This method provides directly the possibility to describe the stationary and non-stationary character of random fields when they are characterized by its spatial covariance function with stationary and non-stationary properties into any region of the field.

On the basis of this method, it is possible to evaluate the reconstruction error surfaces when the location of samples and the spatial covariance function of non-stationary regions are known.

This method presents some contributions to describe the sampling-reconstruction procedure of stationary or non-stationary random fields such as: images or surfaces with varying covariance structure, and the spatial distribution or concentration of substances in the subsurface. Moreover, it is possible to evaluate the error reconstruction of fields or estimate the number and distance of samples in order to provide the reconstruction of fields with a specific quality.

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